International Journal of Theoretical Physics, Vol. 3, No. 1 (1970), pp. 7–21

A Comparison of the Mackey and Segal Models for Quantum Mechanics

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Received: 29 *August* 1969

1. *Introduction*

Mackey (1963) and Segal (1947) have constructed two elegant and quite general models for quantum mechanics. These two models are not equivalent, yet both have led to fruitful results and a deeper understanding of quantum theory. Mackey's work has led to the 'quantum logic' approach which has been carried on by many investigators [for some of these the reader might refer to the bibliographies in Jauch (1968) and Varadarajan (1968)], while Segal's approach is the forerunner of the important C^*-algebra theory of quantum mechanics (e.g., Haag & Kastler, 1964; Segal, 1963). As important as these two models are, very little research seems to have been performed comparing the two. The only works known to the author along these lines are those of Plymen (1967, 1968) and Davies (1968), who have shown that a C^*-algebra can be embedded in a Σ^*-algebra. The C^*-algebra corresponds to the Segal model, and Plymen shows that the Σ^*-algebra satisfies the postulates of Mackey's model. However, a C^*-algebra is much stronger than Segal's original model, and Segal, himself, admits that the distributive and associative laws required in a C^*-algebra are physically very artificial. Also, the Σ^*-algebra cannot be physically justified and is much stronger than the general structure given by what Plymen calls 'Mackey's essential axioms'. In this paper we consider Segal's original model and a generalization of Mackey's model which we feel is physically more reasonable. We then show that the Segal model can be embedded in the Mackey model in a structure-preserving way. This generalizes Plymen's results and shows that Segal's original model can be extended to a Mackey-type model.

2. *The Mackey Model*

We first give a formulation of Mackey's model. Let $\mathcal{O} = \{x, y, z, \ldots\}$ and $M = \{m, m_1, m_2, \ldots\}$ be non-empty collections of objects called *observables* and *states*, respectively, and let $\mathcal{M}$ denote the set of probability measures on the Borel sets $B(R)$ of the real line R.

Axiom 1

There is a map $p:\mathcal{O} \times M \to \mathcal{M}$ denoted by $p(x,m)(\cdot)$.

Axiom 2

If $p(x,m)(E) = p(y,m)(E)$ for every $m \in M$, $E \in B(R)$, then $x = y$. If $p(x,m_1)(E) = p(x,m_2)(E)$ for every $x \in \mathcal{O}$, $E \in B(R)$, then $m_1 = m_2$.

Axiom 3

If $x \in \mathcal{O}$ and f is a real-valued Borel function on R, then there is a $y \in \mathcal{O}$ such that $p(y,m)(E) = p(x,m)(f^{-1}(E))$ for every $m \in M$, $E \in B(R)$.

Axiom 4

If

$$m_1, m_2, \ldots \in M \qquad \text{and} \qquad \sum_{i=1}^{\infty} t_i = 1, 0 \leqslant t_i \leqslant 1$$

then there is an $m \in M$ such that $p(x,m)(E) = \sum t_i p(x,m_i)(E)$ for all $x \in \mathcal{O}$, $E \in B(R)$.

Axiom 5

If E_1, $E_2 \in B(R)$ are disjoint, y and $x \in \mathcal{O}$, and $p(y,m)(E) \geqslant p(x,m)(E_1)$, $p(x,m)(E_2)$ for every $m \in M$, then $p(y,m)(E) \geqslant p(x,m)(E_1 \cup E_2)$ for every $m \in M$.

Axiom 6

If there is an $m \in M$ such that $p(x,m)(E) \neq 0$, then there is an $m_1 \in M$ such that $p(x,m_1)(E) = 1$.

We call a pair $(\mathcal{O}, M)$ satisfying the above six axioms a *weak Mackey model*. Axioms 1, 2, 3, 4 and 6 are identical to Mackey's Axioms I, II, III, IV and VIII respectively. Axiom 5 is weaker than Mackey's Axiom V, which can be expressed in this context as follows:

Axiom 5′

If $x_1, x_2, \ldots$ and $E_1, E_2, \ldots$ are members of $\mathcal{O}$ and $B(R)$, respectively, that satisfy $p(x_i,m)(E_i) + p(x_j,m)(E_j) \leqslant 1$ for all $m \in M$, $i \neq j$, then there exists a $y \in \mathcal{O}$, $F \in B(R)$ such that $p(y,m)(F) = \sum p(x_i,m)(E_i)$ for all $m \in M$.

To show Axioms 1 and 5′ imply Axiom 5, suppose x, y, E, E_1 and E_2 satisfy the hypotheses of Axiom 5. We denote the complement of a set E by E'. Since $p(y,m)(E') = 1 - p(y,m)(E) \leqslant 1 - p(x,m)(E_1), 1 - p(x,m)(E_2)$ we have $p(y,m)(E') + p(x,m)(E_1) \leqslant 1$ and $p(y,m)(E') + p(x,m)(E_2) \leqslant 1$. Applying Axiom 5′, there is a $y_1 \in \mathcal{O}$, $F \in B(R)$ such that

$$1 \geqslant p(y_1,m)(F) = p(y,m)(E') + p(x,m)(E_1) + p(x,m)(E_2) \text{ for all } m \in M.$$

But this can be rewritten $1 \geqslant 1 - p(y,m)(E) + p(x,m)(E_1 \cup E_2)$, which gives Axiom 5. We will give reasons later why we postulate the weaker Axiom 5 instead of Axiom 5′.

Notice that Mackey's Axiom VII, which postulates that the proposition system of quantum mechanics is isomorphic to the lattice of all closed subspaces of a Hilbert space, is not used here. This is because of the 'ad hoc' nature of this axiom. Of the six axioms we have given, Axioms 1, 2 and 5 are the most important, since it is these that give the structure of the proposition system that we shall construct. The other axioms can be dispensed with or weakened in various ways to get a more general theory. However, we shall retain them because they are useful in other contexts and because they can be physically justified.

For the remainder of this section we shall be concerned with constructing and investigating a proposition system which can be associated in a natural way with a weak Mackey model. It follows from Axiom 2 that the observable y in Axiom 3 is unique. We shall denote this observable by $y = f(x)$. It also follows from Axiom 2 that the state m in Axiom 4 is unique, and we denote it by $m = \sum t_i m_i$. We use the notation $m_x(E) = p(x,m)(E)$ and call an observable x a *proposition* if $m_x(\{0,1\}) = 1$ for every $m \in M$. Furthermore, we denote the set of propositions by L. If χ_E is the characteristic function of $E \in B(R)$, we notice that $\chi_E(x) \in L$ for any $x \in \mathcal{O}$. Conversely, if $x \in L$, then x is the characteristic function of an observable; in fact $x = \chi_{\{1\}}(x)$. Notice if $\chi_E(x) = \chi_E(y)$ for all $E \in B(R)$ then $m_x(E) = p(\chi_E(x),m)(\{1\}) = p(\chi_E(y),m)(\{1\}) = m_y(E)$ for all $m \in M$, $E \in B(R)$, and hence $x = y$. Thus the map $E \to \chi_E(x)$ determines the observable x. (This last fact can be used to motivate our later definition of an observable on a proposition system.)

Now suppose $a \in L$, $m \in M$ and $m_a(\{1\}) = s \neq 0$. Then for $E \in B(R)$ we have $m_a(E)$ is 0, 1, s or $1 - s$ depending upon whether $\{1,0\} \not\subset E$, $\{1,0\} \subset E$, $1 \in E$ and $0 \notin E$, or $0 \in E$ and $1 \notin E$, respectively. We thus see that $m_a(\cdot)$ is determined by $m_a(\{1\})$. If we define $m(a) = m_a(\{1\})$ then we obtain a map $m: L \to [0,1]$. Notice that if $m_1(a) = m_2(a)$ for all $a \in L$, then $m_1 = m_2$. We now define for a_1, $a_2 \in L$, $a_1 \leqslant a_2$ if $m(a_1) \leqslant m(a_2)$ for all $m \in M$. It is clear that $\leqslant$ is a partial order relation on L. We define the greatest lower bound $a \wedge b$ and the least upper bound $a \vee b$, if they exist, in the usual way. If f is the identity function $f(\lambda) = \lambda$ on R and $a \in L$, we define the observable a' by $a' = (1 - f)(a)$. Notice that $a' \in L$ and $m(a') = 1 - m(a)$ for all $m \in M$.

If f_0 and f_1 are the functions on R that are identically 0 and 1, respectively, and $x \in \mathcal{O}$, we define the observables 0 and 1 by $0 = f_0(x)$ and $1 = f_1(x)$, respectively. Notice that $0, 1 \in L$ and that 0 and 1 are the unique observable that satisfy $m(0) = 0$ and $m(1) = 1$, respectively, for all $m \in M$. We then have, of course, that $0 \leqslant a \leqslant 1$ for all $a \in L$.

Lemma 2.1

For all a, $b \in L$ the following statements hold. (i) $a'' = a$; (ii) if $a \leqslant b$, then $b' \leqslant a'$; (iii) $a \vee a' = 1$.

Proof: Statements (i) and (ii) are obvious. (iii) Certainly $a, a' \leqslant 1$. Suppose $a, a' \leqslant b$ and $b \neq 1$. Then $b' \neq 0$, and by Axiom 6 there is a state m such that $m(b) = 0$. Then $0 = m(a) = 1 - m(a)$, which is a contradiction. Thus $b = 1$ and $a \vee a' = 1$.

We thus see that L is an orthocomplemented poset. We call the pair (L, M) the *weak logic* of the system. A Boolean sub σ-algebra B of L is *separable* if there is a countable subset D of B such that the smallest Boolean sub σ-algebra containing D is B itself. If $a, b \in L$ satisfy $a \leqslant b'$, we say a and b are *disjoint*, and write $a \perp b$. We say that a Boolean sub σ-algebra B in L is *compatible* if it is separable and if for any mutually disjoint sequence $(a_i) \subseteq B$ and $m \in M$ we have $m(\vee a_i) = \sum m(a_i)$. We will give an example in Section 6 which shows that a separable Boolean sub σ-algebra need not be compatible. We say that a set $P \subseteq L$ is *compatible* if it is contained in a compatible Boolean sub σ-algebra of L. If $a, b \in L$ are compatible we write $a \leftrightarrow b$.

To motivate this definition of compatibility, the propositions may be thought of as corresponding to quantum mechanical events. The compatible Boolean sub σ-algebras correspond to classical probability spaces defined on some phase space. Conversely, if two propositions a and b are physically compatible they correspond to noninterfering events. These noninterfering events interact classically and, hence, should be contained in a classical probability space.

Since a proposition is physically a statement that the measured value of a certain observable is in a certain Borel set, the reader might feel that it would be more appropriate to define compatibility directly in terms of observables. For instance, one might say that $a, b \in L$ are compatible if there is an observable $x \in \mathcal{O}$ and Borel sets E, F such that $a = \chi_E(x)$ and $b = \chi_F(x)$. This gives the physically appealing interpretation that a and b are compatible if, and only if, their validites can be tested by measuring a single observable. However, this is not an adequate definition for compatibility since it may not render compatible certain propositions that should be. This is because there may not be enough observables in $\mathcal{O}$ to give a rich enough supply of compatible propositions. For example, let $\mathscr{B}$ be the set of real Borel functions on R and let $\mathcal{O} = \{f(a) : a \in L, f \in \mathscr{B}\}$. Then $(\mathcal{O}, M)$ is a weak Mackey model. Let $a, b \in L$ be propositions other than 0 or 1. If a and b are compatible according to this latest definition, there must be an $f(c) \in \mathcal{O}(c \in L)$, $E, F \in B(R)$ such that $a = \chi_E(f(c))$ and $b = \chi_F(f(c))$. Then for every $m \in M$,

$$m(a) = p(a, m)(\{1\}) = p(\chi_E(f(c)), m)(\{1\})$$

$$= p(f(c), m)(E) = p(c, m)(f^{-1}(E))$$

We thus see that $f^{-1}(E)$ must contain 0 or 1 but not both, and hence, a is c or c'. Similarly, b is c or c' and hence, either $a = b$ or $a = b'$. Thus a proposition a would be compatible only with 0, 1, a and a'. This is clearly physically unreasonable. More generally, compatibility should not depend in this way on the number of observables in a quantum mechanical system.

The converse of the above should hold, however. That is, if $a = \chi_E(x)$ and $b = \chi_F(x)$ for $x \in \mathcal{O}$, $E, F \in B(R)$, then a and b should be compatible. We will prove this result in Lemma 2.4.

We are now in a position to offer some justification for abandoning Axiom 5′ in favor of Axiom 5. Suppose $a, b \in L$ and $a \leqslant b$. Physically, this means that a has a smaller probability of being true than b. The way these probabilities are determined is by testing the propositions a and b many times and finding the long-run ratio of the number of times they are true to the number of times tested. If $a \leqslant b$ there seems to be nothing in this experimental procedure to justify concluding that a and b can be tested simultaneously with noninterfering experiments, and hence nothing to justify concluding that $a \leftrightarrow b$. In fact, we will show that this does not hold in a system satisfying Segal's postulates for quantum mechanics. However, one can prove that if Axiom 5′ is assumed instead of Axiom 5, then we would have $a \leqslant b$ implies $a \leftrightarrow b$.

Now suppose $a \perp b$. Again considering the definition of the partial order in L, there seems to be no good physical reason why $a \perp b$ should imply $a \leftrightarrow b$, or even that $a \vee b$ exists. This gives us our most compelling reason for not postulating Axiom 5′. Indeed, Axiom 5′ is equivalent to each of the following: if (a_i) is a sequence of mutually disjoint propositions, then (i) $\vee a_i$ exists; (ii) (a_i) is a compatible set in L; (iii) there exists $a \in L$ such that $m(a) = \sum m(a_i)$ for all $m \in M$. However, we show now that Axiom 5 is equivalent to the following more reasonable statement: If (a_i) is a sequence of mutually disjoint propositions such that there is an $x \in \mathcal{O}$ and $E_i \in B(R)$ with $\chi_{E_i}(x) = a_i$, then $\{a_i\}$ is a compatible set in L. For $x \in \mathcal{O}$ we call the set $\{a \in L : a = \chi_E(x) \text{ for some } E \in B(R)\}$ the *range* of x and denote it by $R(x)$.

Lemma 2.2

Let (a_i) be a sequence of mutually disjoint propositions in the range of an observable $x \in \mathcal{O}$ and suppose $\chi_{E_i}(x) = a_i$ for $i = 1, 2, \ldots$. Then

(i) $m_x(E_i \cap E_j) = m(\chi_{E_i \cap E_j}(x)) = 0$ for every $m \in M$, $i \neq j$;

(ii) $m(\chi_{\cup E_i}(x)) = \sum m(\chi_{E_i}(x)) = \sum m(a_i)$ for all $m \in M$.

Proof: (i) Suppose there is an $m \in M$ such that $m(\chi_{E_i \cap E_j}(x)) \neq 0$, $i \neq j$. Then applying Axiom 6 there is an $m_1 \in M$ such that $1 = m_1(\chi_{E_i \cap E_j}(x)) = m_{1x}(E_i \cap E_j)$. But then $m_{1x}(E_i) = m_{1x}(E_j) = 1$ and hence $2 = m_{1x}(E_i) + m_{1x}(E_j) = m_1(a_i) + m_1(a_j)$. This contradicts the fact that $a_i \perp a_j$, $i \neq j$.

(ii) $m(\chi_{\cup E_i}(x))$

$$
\begin{aligned}
&= m_x(\textstyle\bigcup E_i) \\
&= m_x[E_1 \cup (E_2 - E_1 \cap E_2) \cup (E_3 - (E_3 \cap E_1) \cup (E_3 \cap E_2)) \cup \cdots] \\
&= m_x(E_1) + m_x(E_2) - m_x(E_1 \cap E_2) + m_x(E_3) \\
&\quad - m_x((E_3 \cap E_2) \cup (E_3 \cap E_1)) + \cdots
\end{aligned}
$$

It follows from (i) that the negative terms vanish and the lemma follows.

Lemma 2.3

If (a_i) is as in Lemma 2.2, then $\vee a_i$ exists and is given by $\chi_{\cup E_i}(x)$.

Proof: Applying Lemma 2.2 (ii) we have $m(\chi_{\cup E_i}(x)) = \sum m(a_i) \geqslant m(a_j)$ for all j and $m \in M$. Now suppose $b \geqslant a_j$ for all j where $b \in L$. Then $m(b) \geqslant m(a_j)$, $j = 1, 2, \ldots$ and hence $p(b,m)(\{1\}) \geqslant p(x,m)(E_1)$, $p(x,m)(E_2)$. Since by Lemma 2.2 (i) we have $p(x,m)(E_2) = p(x,m)(E_2 - E_1 \cap E_2)$, we can apply Axiom 5 to obtain $p(b,m)(\{1\}) \geqslant p(x,m)(E_1) + p(x,m)(E_2)$.

Now, $p(b,m)(\{1\}) \geqslant p(x,m)(E_1 \cup E_2)$, $p(x,m)[E_3 - (E_1 \cap E_2) \cap E_3]$, and hence

$$p(b,m)(\{1\}) \geqslant \sum_{i=1}^{3} p(x,m)(E_i)$$

By induction we obtain

$$p(b,m)(\{1\}) \geqslant \sum_{i=1}^{n} p(x,m)(E_i)$$

In the limit we have

$$p(b,m)(\{1\}) \geqslant \sum_{i=1}^{\infty} p(x,m)(E_i) = m(\chi_{\cup E_i}(x)$$

and hence $b \geqslant \chi_{\cup E_i}(x)$, which completes the proof.

It follows from the last two lemmas that if (a_i) is a mutually disjoint sequence in the range of an observable, and if $m \in M$, then $m(\vee a_i) = \sum m(a_i)$. We leave the remainder of the proof of the next lemma to the reader.

Lemma 2.4

If $x \in \mathcal{O}$, then the range of x is a compatible subset of L.

3. *Weak Proposition Systems*

Motivated by the considerations of the previous section, we make the following definitions. Let L be an orthocomplemented poset with first and last elements denoted by 0 and 1 respectively. Let M be a set of maps from L into the unit interval $[0,1]$ satisfying: ($L1$) $m(a') = 1 - m(a)$ and $m(1) = 1$ for all $m \in M$, $a \in L$; ($L2$) $a \leqslant b$ in L if an only if $m(a) \leqslant m(b)$ for all $m \in M$; ($L3$) $a \neq 0$ implies the existence of an $m \in M$ with $m(a) = 1$; and ($L4$) if $(m_i) \subset M$ and

$$\sum_{i=1}^{\infty} t_i = 1$$

$0 \leqslant t_i \leqslant 1$, there is an $m \in M$ such that $m(a) = \sum t_i m_i(a)$ for all $a \in L$. The members of L and M are called *propositions* and *states* respectively. A pair (L, M) satisfying ($L1$)–($L4$) is called a *weak proposition system*. We say $P \subseteq L$ is *compatible* in (L, M) if P is contained in a separable Boolean sub σ-algebra B in L such that every state $m \in M$ is additive on disjoint sequences in B. An *observable* x on the weak proposition system (L,M) is a map $x:B(R)\to L$ such that ($\mathcal{O}1$) $x(R)=1$; ($\mathcal{O}2$) the range of x, $R(x)$, is a compatible set in L; ($\mathcal{O}3$) $E\cap F=\varnothing$ implies $x(E)\perp x(F)$; and ($\mathcal{O}4$) if $(E_i)\subseteq B(R)$ is such that $E_i\cap E_j=\varnothing$ for $i\neq j$, then $\vee x(E_i)=x(\bigcup E_i)$. Note that $a\leftrightarrow b$ in L if and only if a and b are in the range of a single observable. If x is an observable and f a real valued Borel function, we define the observable $f(x)$ by $f(x)(E)=x(f^{-1}(E))$ for all $E\in B(R)$. A set of observables $\mathcal{O}$ on (L,M) is said to be *full* if ($F1$) $x\in\mathcal{O}$ implies $f(x)\in\mathcal{O}$ for all real Borel functions f; and ($F2$) if $a\in L$ then there is an $x\in\mathcal{O}$ and $E\in B(R)$ such that $a=x(E)$. Note that every weak proposition system (L,M) supports at least one full set of observables. Indeed, since the set $\{0,a,a',1\}$ is compatible for every $a\in L$, the map $x_a:B(R)\to L$ is an observable when defined by $x_a(E)=a$, a', 0 or 1 according to whether $1\in E$ but $0\notin E$, $0\in E$ but $1\notin E$, $0,\ 1\notin E$ or $0,\ 1\in E$, respectively, and the set $\{f(x_a):a\in L,f\in\mathcal{B}\}$ is full.

Theorem 3.1

Let (L,M) be a weak proposition system and $\mathcal{O}$ a full set of observables on (L,M). Then $(\mathcal{O},M)$ is a weak Mackey model.

Proof: We indicate how each axiom for a weak Mackey model is justified. (1) Define $p:\mathcal{O}\times M\to\mathcal{M}$ by $p(x,m)(E)=m(x(E))$, $E\in B(R)$. (2) If $p(x,m)(E)=p(y,m)(E)$ for all $m\in M$, $E\in B(R)$, then $m(x(E))=m(y(E))$, and using ($L2$) we have $x(E)=y(E)$ for all $E\in B(R)$. Hence $x=y$. Now suppose $p(x,m_1)(E)=p(x,m_2)(E)$ for all $x\in\mathcal{O}$ and $E\in B(R)$. If $a\in L$ then by ($F2$) there is an $x\in\mathcal{O}$ and $E\in B(R)$ such that $x(E)=a$. Hence $m_1(a)=m_2(a)$ for every $a\in L$ and it follows that $m_1=m_2$. (3) This follows from ($F1$). (4) M is closed under convex combinations. (5) If $E_1\cap E_2=\varnothing$ and $p(y,m)(E)\geqslant p(x,m)(E_1)$, $p(x,m)(E_2)$ then $m(y(E))\geqslant m(x(E_1))$, $m(x(E_2))$ for all $m\in M$. Therefore, since M is order determining on L, $y(E)\geqslant x(E_1)$, $x(E_2)$, and hence $y(E)\geqslant x(E_1)\vee x(E_2)=x(E_1\cup E_2)$. It follows that $p(y,m)(E)\geqslant p(x,m)(E_1\cup E_2)$. (6) If $p(x,m)(E)\neq 0$, then $m(x(E))\neq 0$, so $x(E)\neq 0$. Therefore, there is an $m_1\in M$ such that $1=m_1(x(E))=p(x,m_1)(E)$.

This last theorem and the next one show that a weak Mackey model and a weak proposition system are equivalent as far as all of the relevant structure is concerned.

Theorem 3.2

Let $(\mathcal{O},M)$ be a weak Mackey model and (L,M) the associated weak logic. Then (L,M) is a weak proposition system and there is a one-one map τ from $\mathcal{O}$ onto a full set of observables $\mathcal{O}_1$ on (L,M) such that (i) $p(x,m)(E)=m(\tau x(E))$ for all $m\in M$, $x\in\mathcal{O}$ and $E\in B(R)$ (ii) $\tau f(x)=f(\tau x)$ for all $x\in\mathcal{O}$ and real Borel functions f. Furthermore, a subset P of L is compatible in the weak logic (L, M) if and only if it is compatible in the weak proposition system (L, M).

Proof: We have shown that L is an orthocomplemented poset, and it follows easily from the axioms that M has all the necessary properties for (L, M) to be a weak proposition system. Now, if $x \in \mathcal{O}$ we define $\tau x(E) = \chi_E(x)$. To see that τx is an observable on (L, M) note first that by Lemma 2.4 the range of x is a separable Boolean sub σ-algebra such that members of M are additives on disjoint sequences contained in $R(x)$. Therefore, $R(x) = R(\tau x)$ is a compatible subset of the weak proposition system (L, M). It is clear that $\tau x(R) = \chi_R(x) = 1$. Suppose $E \cap F = \varnothing$. Then $E \subset F'$ and by Lemma 2.3 $\chi_{F'}(x) = \chi_{(F' \cap E') \cup E}(x) = \chi_{F' \cap E'}(x) \vee \chi_E(x)$. It follows from Lemma 2.2 that $m(\chi_E(x)) \leqslant m(\chi_{F'}(x))$ for all $m \in M$, and therefore,

$$\tau x(E) \leqslant \tau x(F') = \chi_{F'}(x) = (1 - f) \circ \chi_F(x) = \chi_F(x)' = \tau x(F)'$$

where $f(\lambda) = \lambda$ for all $\lambda \in R$. Thus $\tau x(E) \perp \tau x(F)$. If $E_i \cap E_j = \varnothing$ for $i \neq j$, then $\vee \tau x(E_i) = \vee \chi_{E_i}(x) = \chi_{\cup E_i}(x) = \tau x(\bigcup E_i)$, and it follows that τx is an observable. The remaining details of the proof are left to the reader

Theorem 3.2 was proved by Mackey (1963) for his stronger model and a generalization of his theorem was proved by Maczynski (1967).

The reader will notice that we have departed from the usual procedure of defining propositions to be compatible if they can be split into mutually disjoint propositions, assuming that countable disjoint suprema exist and requiring that states be additive on all disjoint sequences. We have rather taken the set of states M to be a primitive axiomatic concept related to L in the specified way and then defined compatibility in terms of L and M. Physically this approach seems justified, since observables and experimental propositions can only be identified and examined by means of their expectation values in states which are constructed for the given quantum system. We shall give an example later of a weak Mackey model which will show that the usual development given for a partially ordered, orthocomplemented set of experimental propositions is not suitable if the correspondence of Theorem 3.2 is to be preserved.

Let $(\mathcal{O}, M)$ be a weak Mackey model and let $x \in \mathcal{O}$. We define the *expectation* of x in state m by

$$m(x) = \int_R \lambda m_x(d\lambda)$$

if the integral exists. We say x is *bounded* if $\{|m(x)| : m \in M\}$ is bounded and we define the *norm* of x by $|x| = \sup\{|m(x)| : m \in M\}$. We say that an observable z is the *sum* of two bounded observables x and y if $m(z) = m(x) + m(y)$ for all $m \in M$. The sum of two bounded observables need not exist. Note that if τ is the map of Theorem 3.2, then for every $x \in \mathcal{O}$ and $m \in M$ we have $m(x) = m(\tau x)$. Thus τ preserves sums and norms of observables.

4. *The Segal Model*

We now consider the Segal model for quantum mechanics. The observables are the only undefined axiomatic elements in Segal's model. We repeat Segal's axioms for convenience and completeness. A collection of objects X is called a *system of observables* (or *system*, for short) if X satisfies the following postulates.

Axiom A

X is a linear space over the real numbers R.

Axiom B

There exists in X an identity element I and for every $U \in X$ and integer $n \geqslant 0$ an element $U^n \in X$ which satisfies the following: If f, g and h are real polynomials, and if $f(g(\alpha)) = h(\alpha)$ for all $\alpha \in R$, then $f(g(U)) = h(U)$; where

$$f(U) = \beta_0 I + \sum_{k=1}^{n} \beta_k U^k$$

if

$$f(\alpha) = \sum_{k=0}^{n} \beta_k \alpha^k$$

Axiom C

There is defined for each observable U a real number $\|U\| \geqslant 0$ such that the pair $(X, \|\cdot\|)$ is a real Banach space.

Axiom D

$\|U^2 - V^2\| \leqslant \max(\|U^2\|, \|V^2\|)$ and $\|U^2\| = \|U\|^2$.

Axiom E

U^2 is a continuous function of U.

We do not include Segal's Axiom 4 since this axiom has been shown to be redundant by Sherman (1956). A *state* of X is a real valued linear function ω on X such that $\omega(U^2) \geqslant 0$ for all $U \in X$ and $\omega(I) = 1$. A collection of states S on X is *full* if for any two distinct observables U, V there is a state $\omega \in S$ such that $\omega(U) \neq \omega(V)$. Segal (1947) has shown that any system of observables has a full set of states and that $\|U\| = \sup\{|\omega(U)| : \omega \in S\}$ for all $U \in X$. For any two observables U and V the *formal product* $U \circ V$ is defined to be $\frac{1}{4}[(U + V)^2 - (U - V)^2]$. A system is *commutative* if the formal product is associative, distributive (relative to addition) and homogeneous (relative to scalar multiplication). A collection of observables are said to *commute* or form a *commutative collection* if the subsystem generated by the collection is commutative.

Segal (1947) has proved that a commutative system is isomorphic (algebraically and metrically) with the system $C(\Gamma)$ of all real-valued continuous functions on a compact Hausdorff space Γ. The operations in $C(\Gamma)$ are defined in the usual way and the norm is the supremum norm. It is well known that the states on $C(\Gamma)$ consist of the regular Borel probability measures on Γ; that is, if ω is a state, then there is a regular Borel probability measure μ on Γ such that

$$\omega(f) = \int_{\Gamma} f \, d\mu$$

for all $f \in C(\Gamma)$.

An observable $\bigcup \in X$ is an *idempotent* if $\bigcup^2 = \bigcup$. The idempotents correspond to the propositions in the Mackey model. Certainly the observables 0 and I are idempotents, although there may be no other idempotents in the system X. Thus unlike the Mackey model in which the idempotents determine the set of observables, in the Segal model there may be insufficient idempotents to do this. Nevertheless, let us briefly consider the set of idempotents $\mathscr{I}$ in the system X. Since we would like to compare the Mackey and Segal models, it is of interest to study the structure of $\mathscr{I}$. If $a, b \in \mathscr{I}$ we define $a \leqslant b$ if $\omega(a) \leqslant \omega(b)$ for every state ω. It is easy to show that $a \leqslant b$ if and only if $b - a = \bigcup^2$ for some $\bigcup \in X$. If $a \in \mathscr{I}$ it is natural to define $a' = I - a$.

Lemma 4.1

$\mathscr{I}$ is an orthocomplemented poset.

Proof: Clearly $\leqslant$ is a partial order and $0 \leqslant a \leqslant I$ for all $a \in \mathscr{I}$. It is also clear that $a'' = a$ and $a \leqslant b$ implies $b' \leqslant a'$ for all $a, b \in \mathscr{I}$. We now show $a \vee a' = I$ for all $a \in \mathscr{I}$. Suppose $b \in \mathscr{I}$ and $b \geqslant a, a'$. Then $\omega(b) \geqslant \omega(a)$, $1 - \omega(a)$ for all states ω. It follows that $\omega(b) \geqslant \frac{1}{2}$ for all states ω. Now the system generated by b is a commutative system and is thus isomorphic to $C(\Gamma)$ for some compact Hausdorff space Γ. Since b is idempotent, it corresponds to a characteristic function f on Γ. Now Segal (1947) has proved that any state on a subsystem can be extended to a state on the entire system. It follows that $\int f \, d\mu \geqslant \frac{1}{2}$ for every regular Borel probability measure μ on Γ. By considering measures concentrated at points it follows that f is identically one on Γ and hence $b = I$.

We conjecture that $\mathscr{I}$ is actually a lattice. If S is the set of states on X, it follows that $(\mathscr{I}, S)$ is a weak proposition system.

5. *Sherman's Universal Counterexample*

There are two papers of Sherman (1951, 1956) which are of interest to us here. The first paper gives an improvement on Segal's (1947) paper and the second gives an example of a Segal observable system that is useful for counterexamples. As mentioned in Section 2, this example will give further indications of why we must consider weak Mackey models which do not satisfy Axiom 5′.

Let $X = R^3$ and define addition and multiplication by scalars in the usual way. Let $I = (1,1,1)$ and $(\alpha I)^n = \alpha^n I$ for $n \geqslant 0$ an integer. If $x = (x_1, x_2, x_3) \in X$, let $\bar{x} = \max x_i$, $\underline{x} = \min x_i$ and let $X_0 = \{x \in X : \bar{x} = 1, \underline{x} = -1\}$. If $x \in X_0$, define $x^n = x$ if n is an odd integer, and $x^n = I$ if n is an even integer. If $x \in X$, then it is easy to see that there is an $x_0 \in X_0$ such that $x = \alpha x_0 + \beta I$, $\alpha, \beta \in R$. Define

$$x^n = (\alpha x_0 + \beta I)^n = \sum_{j=0}^{n} \binom{n}{j} \alpha^j \beta^{n-j} x_0{}^j$$

It is easy to see that x^n is well defined. For $x \in X$ we define

$$\|x\| = |\bar{x}| = \|\alpha x_0 + \beta I\| = \max\{|-\alpha + \beta|, |\alpha + \beta|\}$$

Sherman (1956) has shown that with these operations X is an observable system. We now consider the set of idempotents $\mathscr{I}$ in X.

Lemma 5.1

If $x \in X$ the following statements are equivalent. (i) $x \in \mathscr{I}$; (ii) $x = \frac{1}{2}(x_0 + I)$, $x_0 \in X_0$; (iii) $x = (x_1, x_2, x_3)$ where $x_i = 1$, $x_j = 0$, $0 \leqslant x_k \leqslant 1$ for i, j, k distinct; $x = I$ or $x = 0$.

Proof: If $x \in \mathscr{I}$, then $x = \alpha x_0 + \beta I = (\alpha x_0 + \beta I)^2 = (\alpha^2 + \beta^2) I + 2\alpha\beta x_0$. If $x_0 = (x_1, x_2, x_3)$, we have $\alpha x_i + \beta = \alpha^2 + \beta^2 + 2\alpha\beta x_i$, $i = 1, 2, 3$. Since $x_i = 1$ and $x_j = -1$ for some i and j, we have $\alpha + \beta = \alpha^2 + \beta^2 + 2\alpha\beta$ and $-\alpha + \beta = \alpha^2 + \beta^2 - 2\alpha\beta$. It follows that $\beta = \frac{1}{2}$ and $\alpha = \pm\frac{1}{2}$. The lemma then follows.

It is easy to see that the order in $\mathscr{I}$ is pointwise order; that is, $x \leqslant y$ if $x_i \leqslant y_i$. Also it is easy to show that $\mathscr{I}$ is a lattice and hence applying Lemma 4.1, $\mathscr{I}$ is an orthocomplemented lattice.

Now it is reasonable to expect that in any structure preserving Mackey-type formulation for X, $\mathscr{I}$ can be embedded in a structure preserving way in the associated proposition system. Thus by examining $\mathscr{I}$ we can get some indication of what is necessary in such a proposition system. We now give an example and a result which show that if the natural structure of $\mathscr{I}$ is to be preserved, then the Mackey model into which X is embedded cannot satisfy Axiom 5′.

We first give an example of two idempotents a, b in $\mathscr{I}$ such that $a \leqslant b$ and yet a and b are not in a Boolean sub σ-algebra of $\mathscr{I}$. We do this by showing that the orthomodular identity $b = a \vee (b \wedge a')$ does not hold. This will also show that $\mathscr{I}$ is not an orthomodular lattice. Let $a = (1, \frac{1}{2}, 0)$ and $b = (1,1,0)$. Then $a \leqslant b$. However, $a \vee (b \wedge a') = (1, \frac{1}{2}, 0) \vee [(1,1,0) \wedge (0, \frac{1}{2}, 1)] = (1, \frac{1}{2}, 0) \vee (0,0,0) = (1, \frac{1}{2}, 0) \neq b$. As was noted in Section 2, if $\mathscr{I}$ is embedded in a Mackey model satisfying Axiom 5′, then $a \leqslant b$ implies $a \leftrightarrow b$.

It is desirable that the two idempotents of this example exist as incompatible propositions in any Mackey-type logic used to reformulate the structure of X, for we can also show that as idempotents in X, a and b do not commute. We do this by showing $2(a \circ b) \neq (2a) \circ b$. Indeed, since $a = \frac{1}{2}(1,0,-1) + \frac{1}{2}I$ and $b = \frac{1}{2}(1,2,-1) + \frac{1}{2}I$, we have $a + b = (1,\frac{1}{2},-1) + I$ and $a - b = \frac{1}{4}(1,-1,1) - \frac{1}{4}I$, and hence
$2(a \circ b) = \frac{1}{2}[(a+b)^2 - (a-b)^2] = \frac{1}{2}[(4,3,0) - (0,\frac{1}{4},0)] = (2,\frac{11}{8},0)$
Similarly, $2a + b = \frac{3}{2}(1,\frac{1}{3},-1) + \frac{3}{2}I$, $2a - b = \frac{1}{2}(1,-1,-1) + \frac{1}{2}I$, and hence $(2a) \circ b = \frac{1}{4}[(2a+b)^2 - (2a-b)^2] = \frac{1}{4}[(9,6,0) - (1,0,0)] = (2,\frac{3}{2},0)$

The next theorem gives further evidence that Axiom 5′ is not acceptable for our purposes.

Theorem 5.1

There is no map $m:\mathscr{I} \to [0,1]$ with $m(I) = 1$ which is additive on all finite disjoint sequences in $\mathscr{I}$.

Proof: Suppose there is such an m. Then $1 = m(I) = m(\vee e_i) = \sum m(e_i)$, where e_i is that member of $\mathscr{I}$ with 1 in the ith entry and 0 elsewhere. Without loss of generality suppose $m(e_1) \neq 0$. Let $\alpha \in R$ be such that $0 < \alpha < 1$ and consider $(0,1,\alpha)$. Note that $e_1 \perp (0,1,\alpha)$ and $e_1 \vee (0,1,\alpha) = 1$. Therefore, $m((0,1,\alpha)') = 1 - m(0,1,\alpha) = m(e_1)$. Also, $m((0,1,\alpha)') = m((1,0,1-\alpha)) = 1 - m(e_2)$ since $(1,0,1-\alpha) \perp e_2$ and $(1,0,1-\alpha) \vee e_2 = 1$. Thus $m(e_1) = 1 - m(e_2)$ or $m(e_1) + m(e_2) = 1$ and it follows that $m(e_3) = 0$. If we replace $(0,1,\alpha)$ by $(0,\alpha,1)$, exactly the same argument shows that $m(e_2) = 0$. It follows, therefore, that $m(e_1) = 1$. But this leads to a contradiction. For again, suppose $0 < \alpha < 1$. Then $m(\alpha,1,0) = 1$, since $(\alpha,1,0) \perp e_3$ and $(\alpha,1,0) \vee e_3 = 1$, while $m(e_1) = 1$ implies $m(e_3) = 0$. Therefore, $m((\alpha,1,0)') = 0$. However, we also have that $m((\alpha,1,0)') = m(1-\alpha,0,1) = 1 - m(0,1,0) = 1$, since $m(e_2) = 0$, $(1-\alpha,0,1) \perp e_2$ and $(1-\alpha,0,1) \vee e_2 = 1$.

6. *Embedding the Segal Model in a Weak Mackey Model*

In this section we consider the problem of embedding any Segal system in a weak Mackey model.

Theorem 6.1

Let X be a Segal system and S its set of states. Then there exists a weak Mackey model $(\mathscr{O},S)$, and a one-one map $\tau: X \to \mathscr{O}$ which satisfies the following conditions: (i) $\tau p(x) = p\tau(x)$ if p is a polynomial; (ii) $\omega(x) = \omega(\tau x)$; (iii) $\tau(x+y) = \tau x + \tau y$; and (iv) $\|x\| = |\tau x|$ for every x, $y \in X$, $\omega \in S$.

Proof: Let $\mathscr{O}_0$ be the set of formal expressions of the form $f(A)$, where A is a commutative subset of X and f is a Borel function on Λ, the compact Hausdorff space such that $C(\Lambda)$ is isometrically isomorphic to the subsystem $X(A)$ of X generated by A. Let $\mathscr{M}$ be the set of probability measures on the Borel sets $B(R)$ of R and define $p_0: \mathcal{O}_0 \times S \to \mathcal{M}$ by $p_0(f(A), \omega)(E) = \mu_{\omega, A}(f^{-1}(E))$ for all $E \in B(R)$ where $\mu_{\omega, A}$ is the regular probability measure on Λ corresponding to ω. If $p_0(f(A), \omega)(E) = p_0(g(B), \omega)(E)$ for all $\omega \in S$, $E \in B(R)$, write $f(A) \cong g(B)$. It is clear that $\cong$ is an equivalence relation. Let $\mathcal{O}$ be the set of equivalence classes and denote the equivalence class containing $f(A)$ by $[f(A)]$. Define $\tau: X \to \mathcal{O}$ by $\tau x = [x]$. If $\tau x_1 = \tau x_2$, then $x_1 \cong x_2$ and $\mu_{\omega, x_1} = \mu_{\omega, x_2}$ for all $\omega \in S$. We then obtain $\omega(x_1) = \omega(x_2)$ for all $\omega \in S$ and since S is full, $x_1 = x_2$. Hence τ is one-one. Define $p: \mathcal{O} \times S \to \mathcal{M}$ by $p([f(A)], \omega)(E) = \mu_{A, \omega}(f^{-1}(E))$ for all $E \in B(R)$. Notice that p is well defined. We now check the six axioms to show $(\mathcal{O}, S)$ is a weak Mackey model. Axiom 1 holds by construction, and Axioms 2 and 4 are easily seen to hold. For $[f(A)] \in \mathcal{O}$ and real Borel function g note that

$$p([g \circ f(A)], \omega)(E) = \mu_{\omega, A}((g \circ f)^{-1}(E)) = \mu_{\omega, A}(f^{-1}(g^{-1}(E)))$$
$$= p([f(A)], \omega)(g^{-1}(E)) = p(g[f(A)], \omega)(E)$$

Therefore, Axiom 3 is satisfied and $g[f(A)] = [g \circ f(A)]$. (6) If $p([f(A)], \omega)(E) \neq 0$, then $\mu_{\omega, A}(f^{-1}(E)) \neq 0$. Let λ be a point in $f^{-1}(E)$ and let μ_λ be the probability measure concentrated at λ. Then μ_λ generates a state on $X(A)$ and by Theorem 4 Segal (1947) (Sherman, 1951, also needed) this state has extension ω_λ to X. Since $\mu_\lambda(f^{-1}(E)) = 1$, $p([f(A)], \omega_\lambda)(E) = 1$. (5) Suppose $E_1, E_2 \in B(R)$ and $E_1 \cap E_2 = \varnothing$. Assume $x, y \in \mathcal{O}$ and $p(y, \omega)(E) \geqslant p(x, \omega)(E_1)$, $p(x, \omega)(E_2)$ for every $\omega \in S$. Without loss of generality we may assume that $x, y \in X$. Let ω be a pure state in $X(x)$. Then, since $\mu_{\omega, x}$ is concentrated at a point, we have $p(x, \omega)(E_1 \cup E_2) = p(x, \omega)(E_1) + p(x, \omega)(E_2)$, where the right-hand side must equal $p(x, \omega)(E_1)$ or $p(x, \omega)(E_2)$. Now again by Theorem 4 Segal (1947), ω has an extension to X and we have $p(y, \omega)(E) \geqslant p(x, \omega)(E_1 \cup E_2)$. Since this inequality is preserved under convex combinations and weak limits, it follows from Segal (1947, p. 940) that $p(y, \omega)(E) \geqslant p(x, \omega)(E_1 \cup E_2)$ for all $\omega \in S$. It is straightforward to show that τ satisfies conditions (i)–(iv) and this is left to the reader. It is also easily seen that (iii) and (iv) follow from (ii).

This last theorem shows that all of the algebraic and metric structure of a Segal system can be recovered in a weak Mackey model.

Corollary 6.2

If $\mathcal{I}$ is the set of idempotents in X and (L, S) the weak logic associated with $(\mathcal{O}, S)$, then for $a, b \in \mathcal{I}$ (i) $a \leqslant b$ in $\mathcal{I}$ if and only if $\tau a \leqslant \tau b$ in (L, S), and (ii) $\tau(a') = (\tau a)'$.

Corollary 6.3

If x and y are commutative in X, then $R(\tau x) \cup R(\tau y)$ is a compatible set in the weak logic (L, S) (where $R(\tau x)$ and $R(\tau y)$ are as in Section 2).

Proof: Since x and y commute, there is a compact Hausdorff space Λ such that $C(\Lambda)$ is isometrically isomorphic to $X(x, y)$. Let $f', g' \in C(\Lambda)$ be such that $[f'(x,y)] = [x]$ and $[g'(x,y)] = [y]$. It is well known that there are real Borel functions h on Λ and f and g on R such that $f' = f \circ h$ and $g' = g \circ h$. Therefore, $[f \circ h(x,y)] = f[h(x,y)] = [x]$ and $[g \circ h(x,y)] = g[h(x,y)] = [y]$. It now follows that $R(\tau x) \cup R(\tau y) \subset R([h(x,y)])$, and Lemma 2.4 implies the corollary.

If (X, S) is a Segal system, we call a triple $(\mathcal{O}, S, \tau)$ satisfying the conditions in Theorem 6.1 a *Mackey realization of* (X, S). If $(\mathcal{O}_1, S)$ and $(\mathcal{O}_2, S)$ are weak Mackey models, we say that $(\mathcal{O}_1, S)$ can be *embedded* in $(\mathcal{O}_2, S)$ if there is a one-one map $\delta: \mathcal{O}_1 \to \mathcal{O}_2$ such that (i) $\delta f(x) = f(\delta x)$ for all $f \in \mathscr{B}$; (ii) $p(\delta x, \omega)(\cdot) = p(x, \omega)(\cdot)$ for all $(x, \omega) \in (\mathcal{O}_1, S)$. We say that $(\mathcal{O}_0, S, \tau_0)$ is a *minimal Mackey realization* of (X, S) if $(\mathcal{O}_0, S, \tau_0)$ can be embedded in any Mackey realization of (X, S).

Theorem 6.4

Any Segal system (X, S) has a minimal Mackey realization $(\mathcal{O}_0, S, \tau_0)$.

Proof: The first part of the proof is similar to the proof of Theorem 6.1. We define $\mathcal{O}_0 = \{[f(x)]: x \in X, f \in \mathscr{B}\}$, and $\tau_0: X \to \mathcal{O}_0$ by $\tau_0 x = [x]$. It follows, as in Theorem 6.1, that $(\mathcal{O}_0, S, \tau_0)$ is a Mackey realization of (X, S). Suppose now that $(\mathcal{O}_1, S, \tau_1)$ is some Mackey realization of (X, S). Define $\delta: \mathcal{O}_0 \to \mathcal{O}_1$ by $\delta[f(x)] = f(\tau_1 x)$. We first show that δ is well defined. Now, it follows from (ii) of Theorem 6.1 that $x \to \omega(x)$ and $x \to \omega(\tau_1 x)$ define the same positive linear functionals and so using the uniqueness of the Riesz representation theorem $p(\tau_1 x, \omega)(E) = \mu_{x,\omega}(E)$ for all $\omega \in M$, $E \in B(R)$. Hence, if $f(x) \cong g(y)$ we have

$$p(f(\tau_1 x), \omega)(E) = \mu_{x,\omega}(f^{-1}(E)) = \mu_{y,\omega}(g^{-1}(E)) = p(g(\tau_1 y), \omega)(E)$$

Therefore $f(\tau_1 x) = g(\tau_1 y)$ and δ is well defined. We now prove properties, (i) and (ii) for an embedding. If $x \in \mathcal{O}_0$ then $x = [g(y)]$ for some $g \in \mathscr{B}$ $y \in X$.

$$\text{(i)}\ \delta f(x) = \delta f([g(y)]) = \delta[f \circ g(y)] = f \circ g(\tau_1 y) = f(g(\tau_1 y))$$
$$= f(\delta[g(y)]) = f(\delta x);$$
$$\text{(ii)}\ p(\delta x, \omega)(E) = p(\delta[g(y)], \omega)(E) = p(g(\tau_1 y), \omega)(E)$$
$$= p(\tau_1 y, \omega)(g^{-1}(E)) = \mu_{y,\omega}(g^{-\prime}(E))$$
$$= p([g(y)], \omega)(E) = p(x_1 \omega)(E).$$

The main disadvantage of the minimal Mackey realization $(\mathcal{O}_0, S, \tau_0)$ is that Corollary 6.3 does not hold, in general, for it.

We now give the example promised in Section 3 to provide justification for our departure from the usual structure given for a system of experimental propositions. Let X be Sherman's example of a Segal system given in Section 5, and let S be its set of states. It follows easily from the proof of

Theorem 6.1 that when $p: X \times S \to \mathscr{M}$ is defined by $p(x, \omega)(E) = \mu_{x,\omega}(E)$ for all $E \in B(R)$, (X, S) is a weak Mackey model and $(\mathscr{I}, S)$ is its weak logic.

First we give an example to show that compatibility in $(\mathscr{I}, S)$ should not be defined in terms of either splitting into disjoint propositions or generating a Boolean sub σ-algebra. Let $a = (1,0,0)$ and $b = (0,1,0)$. It is clear that $a \perp b$ and that a and b generate the eight-element Boolean sub σ-algebra $\{0, (1,0,0), (0,1,0), (0,0,1), (0,1,1), (1,0,1), (1,1,0), I\}$. However, as idempotent observables in X, a and b do not commute. This follows by noting that $X(a,b) = X$ and that X is not commutative. Thus if the notion of 'noninterference' embodied in X is to be preserved in $(\mathscr{I}, S)$, the usual definition of compatibility is not adequate.

Finally, note that the requirement that states on $\mathscr{I}$ be additive on all finite, disjoint sequences is too strong. Indeed, Theorem 5.1 shows that there are no such states on $\mathscr{I}$.

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